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DISCUSSION ON "FREE VIBRATIONS OF BEAMS WITH GENERAL BOUNDARY CONDITIONS"

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The author is to be congratulated for demonstrating mathematically the convergence of using Fourier series method to expand the beam displacement into the superposition of cosine series and an auxiliary polynomial function. The coefficients of the auxiliary polynomial are determined by the general boundary conditions of the Euler–Bernoulli beam under analysis. Rapid convergence and excellent accuracy of eigenfrequencies and modes is achieved [1]. However, some important problems concerned with this paper should be discussed.

1. In the literature survey, the author missed some recent papers where a linear superposition of sine series and an auxiliary polynomial function is applied in the Rayleigh–Ritz method to analyze the vibration of rectangular plates with homogeneous boundary conditions [2] and general boundary conditions [3] respectively. In references [2, 3], the trial functions are developed from the complete solution of a uniform Euler–Bernoulli beam acted upon by a series of static sine loads, which is called as "*static beam functions*".

The general form of the static beam functions is given as follows:

$$w(x) = \sum_{m=1}^{\infty} A_m \sin(\lambda_m x) + c_0 + c_1 x + c_2 x^2 + c_3 x^3, \tag{1a}$$

where $\lambda_m = m\pi/L$, A_m ($m = 1, 2, 3, \ldots$) are the unknown coefficients of the sine series and c_i ($i = 0, 1, 2, 3$) are the coefficients of the auxiliary polynomial, which can be determined by boundary conditions of the beam under consideration. Although references [2, 3] focused on the vibration analysis of rectangular plates, it is obvious that the static beam functions, as given in equation (1a), are also suitable for vibration analysis of the beam with the general conditions.

Equation (1a) can also be rewritten as

$$w(x) = \sum_{m=1}^{\infty} A_m(\sin(\lambda_m x) + c_{m0} + c_{m1} x + c_{m2} x^2 + c_{m3} x^3), \tag{1b}$$

where

$$\sum_{m=1}^{\infty} A_m c_{mn} = c_n, \quad n = 0, 1, 2, 3. \tag{1c}$$

In references [1, 2], the expression of c_{mn} is incorrect and should be replaced by equation (1c), which, however, does not affect the derivations of static beam functions.

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It should be pointed out that if the author expands the beam displacement into Fourier sine series rather than Fourier cosine series, he can similarly gain equation (1a). This means that, in essence, the method of Fourier sine series expansion of beam displacement is the same as the method of static beam functions. Some extensive applications of the method of static beam functions can be found in literature [4–9].

2. In paper [1], the author expanded the beam displacement into the superposition of a Fourier cosine series and an auxiliary polynomial function as follows:

$$w(x) = \sum_{m=0}^{\infty} A_m \cos(\lambda_m x) + \frac{\alpha_1}{24L}(x^4 - 2L^2x^2) - \frac{\alpha_0}{24L}(4L^2x^2 - 4Lx^3 + x^4)$$
$$+ \frac{L^3}{360}(8\alpha_0 + 7\alpha_1) + \frac{\beta_1}{6L}(3x^2 - L^2) + \frac{\beta_0}{6L}(6Lx - 2L^2 - 3x^2), \tag{2a}$$

where α_0, α_1, β_0, β_1 are the unknown coefficients, which can be determined by boundary conditions of the beam. Furthermore, $w(x)$ can be written in form of

$$w(x) = \sum_{m=0}^{\infty} A'_m \cos(\lambda_m x) + b_1x + b_2x^2 + b_3x^3 + b_4x^4, \tag{2b}$$

where

$$A'_0 = A_0 + \frac{L^3}{360}(8\alpha_0 + 7\alpha_1) - \frac{\beta_1 L}{6} - \frac{\beta_0 L}{3}, \quad A'_m = A_m (m = 1, 2, 3, \ldots), b_1 = \beta_0,$$
$$b_2 = \frac{\beta_1}{2L} - \frac{\beta_0}{2L} - \frac{\alpha_1 L}{12} - \frac{\alpha_0 L}{6}, \quad b_3 = \frac{\alpha_0}{6}, \quad b_4 = \frac{\alpha_1 - \alpha_0}{24L}. \tag{2c}$$

It should be pointed out that if the beam is acted upon by a series of cosine static loads as follows:

$$q(x) = \sum_{m=0}^{\infty} P_m \cos(\lambda_m x). \tag{3a}$$

Then, the solution of the beam deflection is

$$w(x) = \sum_{m=0}^{\infty} A'_m \cos(\lambda_m x) + b_1x + b_2x^2 + b_3x^3 + b_4x^4, \tag{3b}$$

where

$$A'_0 = b_0, \quad A_m = P_m/(\lambda_m^4 EI), \quad b_4 = P_0/(24EI). \tag{3c}$$

Equation (3b) can also be rewritten as

$$w(x) = \sum_{m=0}^{\infty} A'_m(\cos(\lambda_m x) + b_{m1}x + b_{m2}x^2 + b_{m3}x^3 + b_{m4}x^4), \tag{3d}$$

where

$$\sum_{m=0}^{\infty} A'_m b_{mn} = b_n, \quad n = 1, 2, 3, 4. \tag{3e}$$

This means that, in essence, the method of Fourier cosine series expansion is also the same as the method of static beam functions of beam displacement.

It should be pointed out that after considering the boundary conditions, the solutions of the static beam functions and the Fourier series expansion are just the same. And the solutions of the beam vibration are also just the same because in such a case, the Ritz solution is the same as the Galerkin solution.

3. The author determined the unknown coefficients α_0, α_1, β_0, β_1 by using

$$H\bar{\alpha} = \sum_{m=0}^{\infty} Q_m A_m, \tag{4a}$$

where

$$\bar{\alpha} = \{\alpha_0, \alpha_1, \beta_0, \beta_1\}^{\mathrm{T}}, \tag{4b}$$

$$H = \begin{bmatrix} \dfrac{8\hat{k}_0 L^3}{360} + 1 & \dfrac{7\hat{k}_0 L^3}{360} & -\dfrac{\hat{k}_0 L^3}{3} & -\dfrac{\hat{k}_0 L}{6} \\ \dfrac{7\hat{k}_1 L^3}{360} & \dfrac{8\hat{k}_1 L^3}{360} + 1 & -\dfrac{\hat{k}_1 L}{6} & -\dfrac{\hat{k}_1 L}{3} \\ \dfrac{L}{3} & \dfrac{L}{6} & \hat{K}_0 + \dfrac{1}{L} & -\dfrac{1}{L} \\ \dfrac{L}{6} & \dfrac{L}{3} & -\dfrac{1}{L} & \hat{K}_1 + \dfrac{1}{L} \end{bmatrix}. \tag{4c}$$

However, for a free–free beam ($\hat{k}_0 = \hat{k}_1 = \hat{K}_0 = \hat{K}_1 = 0$), H becomes

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{L}{3} & \dfrac{L}{6} & \dfrac{1}{L} & -\dfrac{1}{L} \\ \dfrac{L}{6} & \dfrac{L}{3} & -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}. \tag{4d}$$

It is found that the determinant of above matrix H equals zero, i.e.,

$$|H| = 0. \tag{4e}$$

This means that in this case, $\bar{\alpha}$ cannot be determined by equation (4a). For beams with rigid-body displacements, an alternative method has been presented [5, 6].

4. It should be mentioned that Fourier sine series has a better convergence for pinned–pinned beams with rotational restraints, and Fourier cosine series has a better convergence for slid–slid beams with translational restraints. However, for beams with both rotational and translational restraints, we are unable to conclude that Fourier cosine series expansion has better convergence than Fourier sine series expansion or *vice versa*.

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